Concepts in Engineering Mathematics

Part I: Discovery of Number Systems
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Abstract

It is widely acknowledged that *interdisciplinary science* is the backbone of modern scientific research. However such a curriculum is not taught, in part because there are few people to teach it, and due to its inherent complexity and breadth. Mathematics, Engineering and Physics (MEP) are at the core of such studies. To create such an interdisciplinary program, a unified MEP curriculum is needed. This unification could take place based on a core mathematical training from a historical perspective, starting with Euclid or before (i.e., Chinese mathematics), up to modern information theory and logic. As a bare minimum, the *fundamental theorems of mathematics* (arithmetic, algebra, calculus, vector calculus, etc.) need to be appreciated by every MEP student.

At the core of this teaching are 1) partial differential equations (e.g., Maxwell's Eqs), 2) linear algebra of (several) complex variables, and 3) complex vector calculus (e.g., Laplace transforms).

If MEP were taught a common mathematical language, based on a solid training in mathematical history [Stillwell, 2002], students would be equipped to 1) teach and exercise interdisciplinary science and 2) easily communicate with other M, E, and P scientists.

The idea is to teach the history of the development of these core topics, so that the student can fully appreciate the underlying principles. Understanding these topics based on their history (e.g., the people who created them, what they were attempting to do, and their basic mind-set), makes the subject uniformly understandable to every student. The present method, using abstract proofs, with no (or few) figures or physical principles, lacks the intuition and motivation of the original creators of these theories. Such a sterile approach is not functional for many students, resulting in their poor intuition.

Mathematics and its History (MH)

Contents

Mathematics and its History (MH)

WEEK 1

1 Intro+timeline
   3 streams of the Pythagorean theorem
2 Number Systems
3 Integers $\mathbb{Z}$ and Primes $\mathbb{P} \subset \mathbb{Z}^+$

Mathematical Time Line 16-21 CE 1.1.3

1500BCE |0CE |500 |1000 |1400 |1650

Chinese
Babylonia
Pythagoreans
Euclid
Archimedes
Christ
Diophantus
Brahmagupta
Bhaskara
Leonardo
Descartes
Bombelli
Galileo

|1450 |1500 |1600 |1700 |1800 |1900 |1950

Leonardo
Copernicus
Bombelli
Galileo
Mersenne
Fermat
Descartes
Huygens
Newton
Jacob Bernoulli
Johann Bernoulli
Euler
d'Alembert
Daniel Bernoulli
Ben Franklin
Gauss
Cauchy
Riemann
Maxwell
Weierstrass
Abe Lincoln
Hilbert
Poncare
Heaviside
Einstein

Lect[1] NS: 1.1: Introduction: In the beginning . . . 1.1.1

The very first documented mathematics:

- Chinese 5000 BEC (aka BC)
- Babylonians (Mesopotamia/Iraq): integer pairs (a, c), p. 3 1800 BCE
- Pythagorean "triplets" $c^2 = a^2 + b^2$ 540 BEC
- Euclid 300 BEC, Archimedes 287-212 BCE
  - Vol of sphere p. 161, Area of Parabola p. 157
  - Hydrostatics, statics p. 242-4
  - Geometric series p. 182
- Euclid in Alexandria during the reign of Ptolemy I
  - Egypt founded by Alexander the Great 322 BEC
  - Euclid's *Elements* is (was?) the most influential works of mathematics.
  - Geometry every student is assumed to learn in High School
- Alexandria Library burned $\rightarrow$ 'All recorded knowledge destroyed' 391 CE

[1] Part Number-Systems: Lecture.Week

Chronological history of mathematics, by century 1.1.2

500^{th} BCE Chinese (quadratic equation)

180^{th} BCE Babylonia (Mesopotamia/Iraq) (quadratic equation)

6^{th} BCE Py·thag′·o·re·an & tribe 580-500 BCE; Diophantus ≈200-285 BCE

4^{th} BCE Euclid 300 CE, Archimedes 287-212 CE

7^{th} CE Brahm′·a·gup·ta (negative numbers; quadratic equation)

15^{th} Copernicus 1473-1543 Renaissance mathematician & astronomer

16^{th} Tartaglia (cubic eqs); Bombelli (complex numbers); Galileo

17^{th} Newton 1642-1727 *Principia* 1687; Mersenne; Huygen; Pascal; Fermat, Descartes (analytic geometry); Bernoullis Jakob, Johann & son Daniel

18^{th} Euler 1748 *Student of Johann Bernoulli*; d'Alembert 1717-1783; Kirchhoff; Lagrange; Laplace; Gauss 1777-1855

19^{th} Möbius, Riemann 1826-1866, Galois, Hamilton, Cauchy 1789-1857, Maxwell, Heaviside, Cayley

20^{th} Hilbert ($\mathbb{R}$); Einstein; ...

Pythagorean Theorem Road Map: $\Rightarrow$ Three "Streams" 1.1.4

- The Pythagorean Theorem (PT) is the *cornerstone of mathematics*
  - PT is the mathematical fountain for these 3 streams
- $\approx$Five(?) century per stream:
  1) Numbers:

     6^{th}BC $\mathbb{Z}^+ \equiv \mathbb{N}$ (positive integers), $\mathbb{Q}$ *(Rationals)*
     5^{th}BC $\mathbb{Z}$ *Int*, $\mathbb{J}$ *irrationals*
     7^{th}c $\mathbb{Z}$ zero
  2) Geometry (e.g., lines, circles, spheres, toroids, ...)

     17^{th}c Composition of polynomials (Descartes, Fermat)
     Euclid's Geometry + algebra $\Rightarrow$ Analytic Geometry
     18^{th}c Fundamental Thm of Algebra
  3) Infinity ($\infty \rightarrow$ Sets)

     17-18^{th}c $\mathcal{F}$ *Taylor series, Functions, Calculus* (Newton)
     19^{th}c $\mathbb{R}$ *Real*, $\mathbb{C}$ *Complex* 1851
     20^{th}c Set theory

NS Lect 2.1 *Stream 1:* Number systems c1000 CE Ch. 3 2.1.1

Integers $\mathbb{Z}^+, \mathbb{Z}$ and rationals $\mathbb{Q}$ were the only "legal" numbers:

- Brahmagupta 628 CE used zero, and *negative integers* $\mathbb{Z}_-$ for debt.
- 9^{th} century: the symbol 0 entered the Arabic number system
- *Cardinal numbers* 5000 CE: Birds & Bees "count" cardinality
- *Positive Integers* $\mathbb{Z}_+$, Integers $\mathbb{Z}$
- *Rational numbers* $\mathbb{Q}$: Egyptians c1000 CE; Pythagoras 500 CE
- *Prime numbers*
  - Fundamental Theorem of Arithmetic: p. 43;
  - *Euclid's formula* for *Pythagorean Triplets*;
  - *Greatest Common prime Divisor*; *Euclidean algorithm* p. 41
    Exs: (15=3*5, 30=2*3*5): gcd=5, lcd=3
  - Prime number Theorem

Fundamental theorems for primes p. 43 2.1.2

- Fundamental theorem of Arithmetic:

  Every integer n may be written as a product of primes.

  Every integer N may be written as a product of primes π_k, of multiplicity m_k

  $$N = \Pi_k \pi_k^{m_k}$$

- Examples:
  - $27 = 3^3$; $6 = 2 \cdot 3$; $297 = 3^3 \cdot 11$
  - $1001 = 7 \cdot 11 \cdot 13$
  - $14 = \pi_1 \cdot \pi_4$
  - $28 = 7 \cdot 2 \cdot 2 = \pi_4 \cdot \pi_1^2$
  - $3881196 = 2^2 \cdot 11^3 \cdot 27^2 = \pi_1^2 \cdot \pi_2^6 \cdot \pi_5^3$

Fundamental theorems for primes p. 43

- *Prime number Theorem:* Density of primes[2]

$$\pi(N) \equiv \sum_{2}^{N} \delta(\pi_k) \approx Li(N) \equiv \int_2^N \frac{d\xi}{\ln(\xi)}$$

where $\delta(k) = 1$ if k is a prime and zero otherwise.

> *Chebyshev said, and I say it again. There is always a prime between n and 2n.* *p. 585*$_3$

Namely

$$\pi_{N+1} - \pi_N \propto \ln(N)$$

[2] http://en.wikipedia.org/wiki/Prime_number_theorem

Stream 2: Geometry Ch 2 & p. 37-8 2.1.4

- Roles of Number theory vs. Geometry in Mathematics p. 38
  - Geometry is stabilizing and unifying
  - Number theory spur to progress and change
  - The *Fundamental Thm of Algebra*

Fundamental Theorem of Algebra 2.1.5

Stream 2 defines early analytic geometry:

> *Every polynomial equation $p(z) = 0$ has a solution in the complex numbers. As Descartes observed, a solution $z = a$ implies that $p(z)$ has a factor $z - a$. The quotient*
>
> $$q(z) = \frac{p(z)}{z - a}$$
>
> *is then a polynomial of one lower degree. . . . We can go on to factorize $p(z)$ into n linear factors.*

An early observation on complex roots:

> *. . . d'Alembert (1746) observed that for polynomials $p(z)$ with real coefficients, if $z = u + iv$ is a solution of $p(z) = 0$, then so is its conjugate $z^* = u - iv$. Thus the imaginary linear factors of a real $p(z)$ can always be combined in pairs with real coefficients.*

Stream 3: Infinity and irrational numbers Ch 4 2.1.6

- Limit points, open vs. closed sets are fundamental to modern mathematics
- These ideas first appeared with the discovery of $\sqrt{2}$, and $\sqrt{n}$
  https://en.wikipedia.org/wiki/Spiral_of_Theodorus
  and related constructions (factoring the square, Pell's Eq. p. 44)

Let $A(x)$ be the area under $f(x)$. Then

$$\frac{d}{dx}A(x) = \frac{d}{dx}\int^{x} f(\eta)d\eta$$
$$= \lim_{\delta\to 0}\frac{A(x+\delta)-A(x)}{\delta}$$

and/or

$$A(b)-A(a) = \int_a^b f(\eta)d\eta$$

- Stream 3 is about limits
- Integration and differentiation (Calculus) depend on limits
- Limits are built on open vs. closed sets

Lect NS 3.1 *Stream 1:* Integers and Primes 3.1.1

- Roman number system
- The first abacus (Romans introduced concept to Chinese)
- The positive integers $\mathbb{Z}_+$
- The first use of zero (Brahmagupta 628 CE)
- Rational numbers $\mathbb{Q}$

Why were/are integers important? 3.1.2

- Pythagorean Motto: *All is number*
- Integers were linked to Physics: i.e., Music and Planetary orbits
- Today:

  > *With the digital computer, digital audio, and digital video coding everything, at least approximately, [is transformed] into sequences of whole numbers, [thus] we are closer than ever to a world in which "all is number."* *p. 16*

  - Public-private key encryption is based on factoring large integers (very hard)
  - Quantum Mechanics (quantization of states)
- The identification of irrational numbers $\mathbb{Q} \subset \mathbb{R}$ spoiled the concept of integer perfection

Beauty of integers: Aristotle, Pythagoras 3.1.3

- Strings
- Chinese Bells & chimes
  - The Physics and math of musical instruments
  - Acoustic Transmission lines
  - Eigen-modes: Mathematics in Music and acoustics:
    - Guitar strings
    - Bells
    - Tuning forks
    - Organ pipes
  - Damping of eigen-modes $\rightarrow$ complex eigen-modes
- Quantum states as normal modes
  - Electrons in a "box"
  - Radiation is a form of damping

Pythagoreans 3.1.4

- The Pythagoreans lived in Croton (Southern Italy)
  - "Musical principles played almost as important a part in the Pythagorean system as mathematical or numerical ideas." –Wikipedia
  - The Octave (2x in frequency); Perfect Third & Fifth; Harmonics
- Vincenzo Galilei (father of Galileo) is known to have "discovered a new mathematical relationship between string tension and pitch . . . which paved the way to his son's crucial insight that all physical phenomena – leading to modern physics"
- The Pythagoreans were the first to investigate musical scales as rational numbers, particularly of small integers (e.g., 3/2).
- Pythagoreans' central doctrine:
  "The physical world arises from the harmony of whole numbers."[3]

[3] https://en.wikipedia.org/wiki/Music_and_mathematics

Small but significant error in scales

- The Pythagoreans argued that 12 *perfect fifths* ($\frac{3}{2}$) is 7 octaves

$$\left(\frac{3}{2}\right)^{12} \neq 2^7 = 128$$

- This ratio is in error by a significant 1.3% (129.745/128)
- This argument is based on their belief in integral musical relationships.
- How to count musical half-steps, spaning a musical fifth:

$$C, C^{\#}, D, D^{\#}, E, F, G$$

- Today, each half-step is $\sqrt[12]{2}$: $\Rightarrow$ 12 steps is one octave (the *well-tempered* scale)

BCE Pythagoras; Aristotle; stringed instruments and Integers

16^{th} Mersenne, Marin 1588-1647; *Harmonie Universelle 1636*, *Father of acoustics*; Galilei, Galileo, 1564-1642; *Frequency Equivalence 1638*

17^{th} Properties of sound: Newton, Sir Issac 1686; First calculation of the speed of sound; Hooke, Robert; Boyle, Robert 1627-1691;

18^{th} Waves and Thermodynamics: Bernoulli, Daniel (#3); Euler; Lagrange; d'Alembert;

19^{th} Math, Sound and Electricity Gauss; Riemann; Laplace; Fourier; Helmholtz; Heaviside; Bell, AG; Rayleigh, Lord (aka: Strutt, William)

20^{th} Math and communication theory: Hilbert, David; Cambpell, George Ashley; Noether, Emmy; Fletcher, Harvey; Bode, Henrik; Nyquist, Harry; Dudley, Homer; Shannon, Claude;

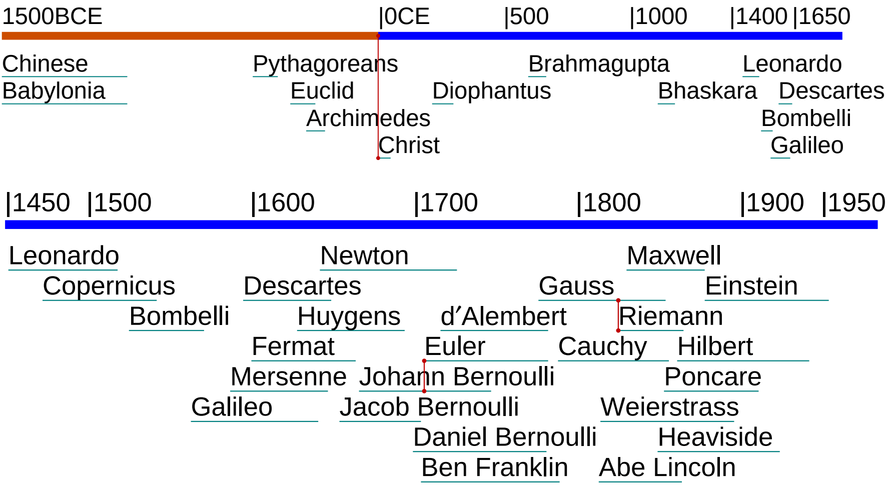
WEEK 2

4 Primes $\pi_k \in \mathbb{P}$

5 Pythagorean triplets $\{a, b, c\}$

6 Greek number theory & GCD(a,b)

Mathematical Time Line 16-21 CE 1.1.3



NS Lect 4.2 *Stream 1:* Prime numbers π_k 4.2.1

- Every prime has only two factors: 1 and itself
- $\pi_k|_{k=1}^{\infty} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 \ldots\}$
- Are most primes odd? All primes but 2 are odd
- How many primes are divisible by 3? all but 3
- Mersenne Primes: $2^n - 1$ ($2^n - 1$ is not prime for $n = 4, 6, 8, 9, 10$)
  - What n's give Mersenne Primes? n must be prime.
- *Fundamental theorem of arithmetic: Every integer N may be written as a product of primes π_k, of multiplicity m_k*

$$N = \Pi_k \pi_k^{m_k}$$

- Examples:
  - $14 = \pi_1 \cdot \pi_4$
  - $28 = 7 \cdot 2 \cdot 2 = \pi_4 \cdot \pi_1^2$
  - $3881196 = 2^2 \cdot 11^3 \cdot 27^2 = \pi_1^2 \cdot \pi_2^6 \cdot \pi_5^3$
- Gaussian primes: $3 + 7j$
- How to find primes? Sives

Fundamental theorem of Arithmetic p. 43 4.2.2

Every integer n may be written as a product of primes.

- Examples: $27 = 3^3$; $6 = 2 \cdot 3$; $297 = 3^3 \cdot 11$
- Primes may be identified using a "sieve"

- *Sieve of Eratosthenes:*
  - Start with a list of N "elements" $n_k := \{2, \cdots, N\}$
  - The first list element n_1 is the next prime π_k
  - Highlight all $n_k := \pi_k \cdot n_k$
  - Delete all the highlighted elements
  - Repeated on the reduced n_k

Figure: Sieve of Eratosthenes:

- *Euler's sieve:*

  https://en.wikipedia.org/wiki/Sieve_of_Eratosthenes#Euler.27s_Sieve

Greatest common factors GCD(n,m) 4.2.4

- The GCD of two numbers is the largest common factor
- If you factor the two numbers the GCD is "Obvious"
- Examples:
  - gcd(13,11) $=$ 1 The gcd of two primes is always 1
  - gcd(13*5,11*5) $=$ 5 The common 5 is the gcd
  - gcd(13*10,11*10) $=$ 10 The gcd(130,110) $=$ 10 $=$ 2*5, is not prime
  - gcd(1234,1024) $=$ 2 (1234$=$2*617, 1024$=2^{10}$)
- *Co-primes* ($a \perp b$) are numbers with no common factors (but 1)
  - Example: $a = 7 * 13, b = 5 * 19 \Rightarrow (7 * 13) \perp (5 * 19)$
  - I.E.: If $a \perp b$ then $gcd(a, b) = 1$
- $a/\gcd(a, b) \in \mathbb{Z}$

Riemann Zeta Function $\zeta(s)$ 4.2.4

- Integers appear is the "roots" (aka eigenmodes) of $\zeta(s)$
- Basic properties ($s = \sigma + i\omega$)

$$\zeta(s) \equiv \sum_{1}^{\infty} \frac{1}{n^s} \qquad \sigma = \Re(s) > 0$$

  - What is the *region of convergence (ROC)?*
- The amazing *Euler-Riemann Product formula*:

$$\zeta(s) = \prod_k \frac{1}{1 - {\pi_k}^{-s}} = \prod_k \frac{1}{1 - \left(\frac{1}{\pi_k}\right)^s} = \prod_k \frac{1}{1 - \frac{1}{\pi_k^s}}$$
$$= \frac{1}{1-2^{-s}} \cdot \frac{1}{1-3^{-s}} \cdot \frac{1}{1-5^{-s}} \cdot \frac{1}{1-7^{-s}} \cdots \frac{1}{1-p^{-s}} \cdots$$

- Euler c1750 assumed $s \subset \mathbb{R}$. Riemann c1850 extended $s \subset \mathbb{C}$

Plot of $|\zeta(s)|$

Angle of Riemann Zeta function $\angle\zeta(z)$ as a function of complex z

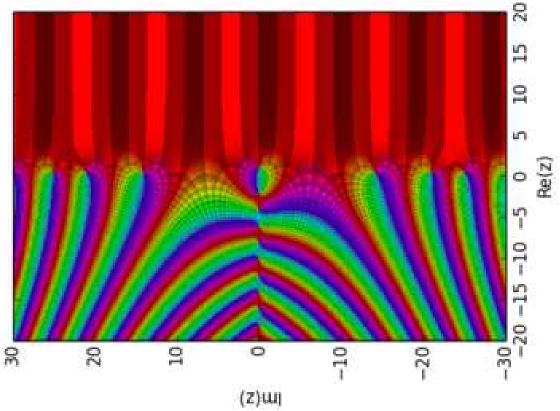


Figure: $\angle\zeta(z)$: Red $\Rightarrow \angle\zeta(z) < \pm\pi/2$

Physical meaning of each factor of $\zeta(s)$ 4.2.5

- Series expansion

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \text{ROC: } |x| < 1$$

- If time T is a positive delay, then from the *Laplace transform*

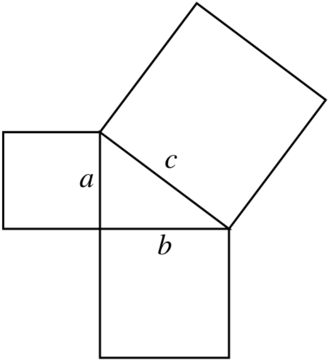
$$\delta(t-T) \leftrightarrow \int_0^\infty \delta(t-T)e^{st}dt = e^{-sT}$$

- Each factor of $\zeta(s)$ is an ∞ sum of delays
- For example for $\pi_1 = 2$, ($T = \ln(2)$, thus $2^{-2} = e^{-s\ln 2}$)

$$\sum_n \delta(t-nT) \leftrightarrow \frac{1}{1-2^{-s}} = 1 + e^{-sT} + e^{-s2T} + \cdots$$

Lect NS 5.2 Pythagorean triplets $c^2 = a^2 + b^2$ p. 43 5.2.1

- Pythagoras assumed that $[a, b, c] \subset \mathbb{Z}$ (i.e., are integers)
- This relationship has a deep meaning and utility



Proof of Pythagoras's Theorem

Pythagorean triplets: $b = \sqrt{c^2 - a^2}$ 5.2.2

Applications in architecture and scheduling (quantized units)

EXERCISES

The integer pairs (a, c) in Plimpton 322 are

a	c
119	169
3367	4825
4601	6649
12709	18541
65	97
319	481
2291	3541
799	1249
481	769
4961	8161
45	75
1679	2929
161	289
1771	3229
56	106

Figure 1.3: Pairs in Plimpton 322

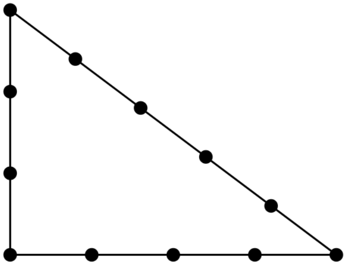
1.2.1 For each pair (a, c) in the table, compute $c^2 - a^2$, and confirm that it is a perfect square, b^2. (Computer assistance is recommended.)

You should notice that in most cases b is a "rounder" number than a or c.

1.2.2 Show that most of the numbers b are divisible by 60, and that the rest are divisible by 30 or 12.

Unit-length jointed sides (e.g., bricks) 5.2.3

Integer property of Pythagoras's Theorem:

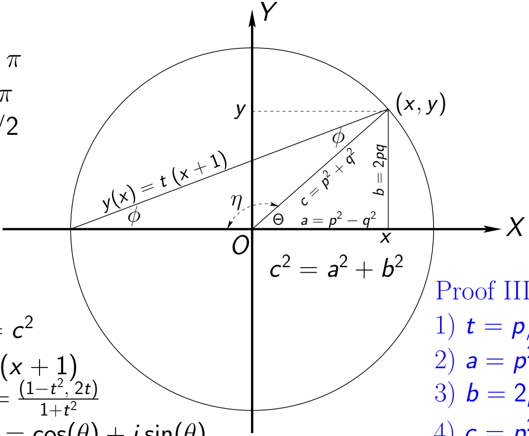


- Pythagoras required that the sides are $\subset \mathbb{Z}^+$ (e.g.: [3,4,5])
- Note that $3 = \sqrt{4+5}$ (i.e., $a = \sqrt{b+c}$)
- Why?

Euclid's Formula for Pythagorean triplets $[a, b, c]$

Proof I:

1) $2\phi + \eta = \pi$
2) $\eta + \Theta = \pi$
3) $\therefore \phi = \Theta/2$



Proof II:

1) $x^2 + y^2 = c^2$
2) $y(x) = t\,(x+1)$
3) $\therefore (x, y) = \frac{(1-t^2,\, 2t)}{1+t^2}$
4) $e^{i\theta} = \frac{1+it}{1-it} = \cos(\theta) + i\sin(\theta)$

Proof III:

1) $t = p/q$
2) $a = p^2 - q^2$
3) $b = 2pq$
4) $c = p^2 + q^2$

Choose $p, q, N \subset \mathbb{Z}_+$, with $p = q + N$, then $c^2 = a^2 + b^2$ (p. 8)

Exp: $N = 1, q = 1 : p = 2$: $a = 2^2 - 1^2 = 3$, $b = 2 \cdot 1 \cdot 2 = 4$, $c = \sqrt{3^2 + 4^2} = 5$

- Step 4 of Proof II was not shown for lack of room on the slide
  Define (x, y) in terms of *complex number* $\zeta \equiv x + iy$
- Since $\zeta(\Theta)$ lies on the unit circle ($|\zeta| = 1$)

$$\zeta(\Theta) = e^{i\Theta} = \cos(\Theta) + i\sin(\Theta)$$

$$e^{i\Theta(t)} = \frac{1 - t^2 + i2t}{1 + t^2} = \frac{(1 + it)(1 + it)}{(1 + it)(1 - it)} = \frac{(1 + it)}{(1 - it)}$$

- The Greeks were looking for Pythagorean Triplets $[a, b, c] \in \mathbb{Z}^+$:
  - They composed a line II.2 & circle II.1.
  - Solving for $\zeta(\Theta)$ gives formulas for $[a, b, c]$, and
    a) the quadratic equation, b) complex numbers, c) Euler's formula
  - Only today can we fully appreciate all these details.

Whats going on? 5.2.6

- Assume that we have a glass bead on a wire circle at a position that depends on the angle with τ as the time

$$\Theta(\tau) = 2\pi\tau$$

- The bead traverses the wire every second with a circular motion

$$e^{i\Theta(\tau)} = e^{i2\pi\tau} = \cos(2\pi\tau) + i\sin(2\pi\tau)$$

- The real and imaginary parts define a sinusoid (i.e., a pure tone)
- The function e^{τ} is the *eigen-function* of *operator* $d/d\tau$

$$\frac{d}{d\tau}e^{a\tau} = ae^{a\tau}$$

  - Thus $e^{a\tau}$ is the key when solving differential equations

Whats going on? 5.2.7

- Assume that we have a glass bead on a wire circle at a position that depends on the angle
$$\Theta(t) = 2\pi\tau$$
where τ is time.
- Then the bead will traverse the wire every second.
- We may invert $\Theta(t)$ to find $t(\Theta)$

$$1 + it = (1 - it)e^{i\Theta}$$
$$it\left(e^{i\Theta} + 1\right) = e^{i\Theta} - 1$$
thus:
$$it(\Theta) = \frac{e^{i\Theta} - 1}{e^{i\Theta} + 1} = \frac{e^{i\Theta/2} - e^{i\Theta/2}}{e^{i\Theta/2} + e^{i\Theta/2}} = i\tan(\Theta/2)$$

- The slope of the line t varies as $t = \tan(\pi\tau)$ (i.e., wildly)
- This example shows the natural simplification of polar coordinates

Pythagoras 5.2.8

- Pythagorean "band" (i.e., a team of fellow mathematicians)
  - Based on integer relations
  - Work for hire
  - Discovery of $\sqrt{2}$ "story"
- Pythagorean were destroyed by superstition (i.e., ignorance + fear):

  > *Whether the complete rule of number (integers) is wise remains to be seen. It is said that when the Pythagoreans tried to extend their influence into politics they met with popular resistance. Pythagoras fled, but he was murdered in nearby Metapontum in 497 BCE.* *p. 16*

- The relations between the integers were assumed to be a reflection of the physical world: *All is number*
- This view followed from the musical scale and other observations
- However the view broke given the diagonal of a unit-square:

$$d = \sqrt{1^2 + 1^2} = \sqrt{2} \tag{1}$$

- Question: Is $\sqrt{2} \subset \mathbb{Q}$?
  - They soon "proved" $\sqrt{2}$ was not *rational*, thus
  - It was termed *irrational* $\not\mathbb{Q}$ (*not* a ratio of two integers)

Taxonomy of numbers 6.2.2

The numerical hierarchy (i.e., "taxonomy"): From primes ... complex

- $\pi_k \in \mathbb{P} \subset \mathbb{Z}^+ \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{J} \subset \mathbb{R} \subset \mathbb{C}$
- *Rational numbers* $\mathbb{Q}$ subset of *irrational numbers* $\mathbb{J}$: $\mathbb{Q} \subset \mathbb{J}$
- Theoredoris Spiral of $\sqrt{n}$ generates all $\sqrt{n}, n \in \mathbb{Z}^+$
  https://en.wikipedia.org/wiki/Spiral_of_Theodorus
  - GCD with $(n+1)^2 = n^2 + 1^2$, $n = 1 \cdots$
  - Continued Fractions (extended Euclidean algorithm)
  - Factoring the *golden rectangle* $1 \times (1 + \sqrt{2})$
- The first applications of $0, \infty$ (Brahmagupta $7^{th}c$)
- European text on algebra of negative numbers (Bombelli $16^{th}c$)
- First calculations with $\sqrt{-1}$ (Bombelli $16^{th}c$); first accepted (1851)
- Fields (functions of several variables: $\phi(t, x, \cdots)$)
- Vectors (dimensional groups of fields: $\boldsymbol{E}(x, y, z, t) = [E_x, E_y, E_z]^T$)

Sprial of Theodorus

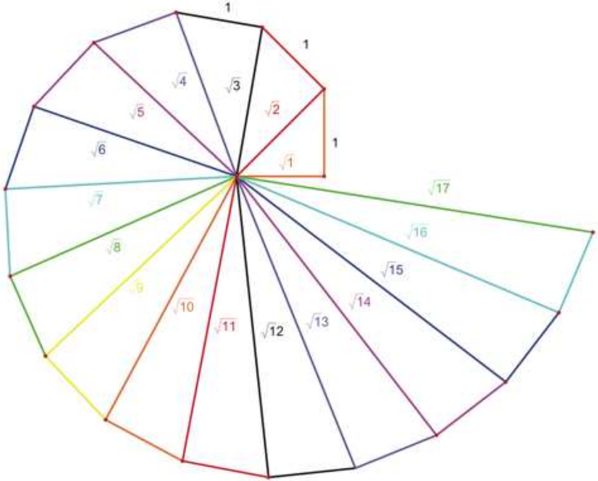


Figure: Sprial of Theodorus: $\sqrt{n+1}^2 = \sqrt{n}^2 + 1^2$

Euclid's algorithm (GCD) 6.2.4

Greatest common divisor GCD of two integers p. 42:

- Let $a_0 = N$, $b_0 = M$
- Recurse on index $k = 0, 1, 2, \ldots$

$$a_{k+1} = \max(a_k, b_k) - \min(b_k, b_k) \tag{2}$$
$$b_{k+1} = \min(a_k, b_k) \tag{3}$$

- When $a_k = b_k$ the GCD is a_k

$$\begin{aligned}
a_0 &= 30 = 2 \cdot 3 \cdot 5 & b_0 &= 35 = 5 \cdot 7 \\
a_1 &= (7 - 2 \cdot 3) \cdot 5 = 5 & b_1 &= (2 \cdot 3) \cdot 5 = 30 \\
a_2 &= (2 \cdot 3 - 1)5 = 25 & b_2 &= 5, \quad \cdots \\
a_k &= 20, 15, 10, 5 & b_k &= 5, \quad k = 3, 4, 5, 6 \\
a_6 &= (2 - 1)5 = 5 & b_5 &= 5 \to \textit{done}. \quad \therefore GCD = 5
\end{aligned}$$

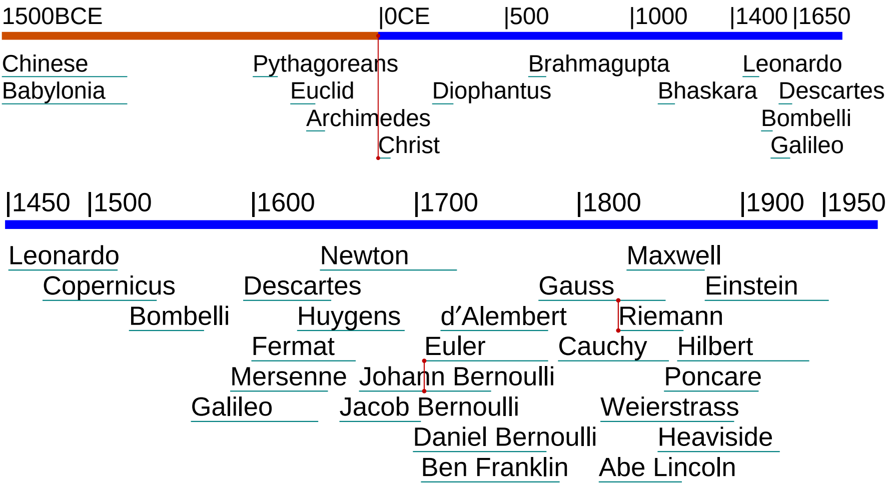
NOTE:GCD(25,75)=25 $= 5^2$, not 5: It returns π_k^m

Discuss Euclid's Algorithm (GCD) vs. formula (PT) 6.2.5

- Euclid's *formula* (PT $[a, b, c]$)
- Euclid's *Algorithm* (GCD)
  - See Page 47_3 (Problem 3.4.1)
    GCD with subtraction replaced by division leads to the *continued fraction algorithm*

WEEK 3

- * Labor Day
- 7 *Continued fractions* & the GCD (e.g., $\pi \approx 22/7$)
- 8 Geometry and irrational numbers: $\sqrt{n} \in \mathbb{Q} \in \mathbb{R}$
  Approximating roots of $\sqrt{5}$
  The *Fibonacci sequence* and its difference equation


1500BCE
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Babylonia
Pythagoreans
Euclid
Archimedes
Christ
Diophantus
Brahmagupta
Bhaskara
Leonardo
Descartes
Bombelli
Galileo
|1450
|1500
|1600
|1700
|1800
|1900
|1950
Leonardo
Copernicus
Bombelli
Descartes
Fermat
Mersenne
Galileo
Newton
Huygens
Johann Bernoulli
Jacob Bernoulli
d'Alembert
Euler
Daniel Bernoulli
Ben Franklin
Gauss
Cauchy
Weierstrass
Abe Lincoln
Maxwell
Riemann
Einstein
Hilbert
Poncare
Heaviside

Lect 7.3 Stream 2: Continued Fraction Algorithm 7.3.1

- A variation on the GCD is the *Continued Fraction Algorithm* CFA
- CFA is useful for finding rational approximations
- The CFA proceeds as the GCD, but with very different steps:
  - Given any starting number α: For $k = 1, \ldots; \alpha_0 = \alpha$:
    1) $n_k = round(\alpha_{k-1})$
    2) $\alpha_k = 1/(\alpha_{k-1} - n_k)$; *next k*
- Given a rational α, CFA terminates:
  - Examples: $\pi \approx 22/7 = (21+1)/7 = 3 + 1/7$
  - $\pi \approx \frac{355}{113} \Rightarrow 1/\left(\frac{355}{113} - 3\right) = 7.065 = 7 + 1/16$
- For irrational α (i.e, π) CFA does not terminate

Rational approximations: Continued fraction expansion of π 7.3.2

- Example of expanding π:

$$\pi = 3 + (\pi - 3) = 3 + \frac{1}{{\color{red}7}.0625}$$

$$= 3 + \cfrac{1}{{\color{red}7} + \cfrac{1}{16 + \cfrac{1}{-294 + \cfrac{1}{\cdots}}}} \quad =: \mathbf{3 + 1 \int 7 + 1 \int 16 + 1 \int -294 +}$$

- Examples:
  - $\frac{22}{7} = 3 + 1/7 \approx \pi + O(1.3 \times 10^{-3})$
  - $\frac{355}{113} = 3 + 1 \int 7 + 1 \int 16 \approx \pi + O(2.7 \times 10^{-7})$
  - $\frac{104348}{33215} = 3 + 1 \int 7 + 1 \int 16 + 1 \int -294 \approx \pi + O(3.3 \times 10^{-10})$
- Which is better, to *fix*() or to *round*()?

Prime Rational approximations: 7.3.3

- Reciprocate and subtract out the nearest prime:

$$
\begin{aligned}
\pi &= 3 + (\pi - 3) = 3 + \frac{1}{7.0625} \\
&= 3 + \cfrac{1}{7 + \cfrac{1}{{\color{blue}16 \to 17} + \cfrac{1}{\pi_? + \cfrac{1}{\cdots}}}} \quad =: \mathbf{3} + \mathbf{1} \int \mathbf{7} + \mathbf{1} \int \mathbf{17} + \mathbf{1} \int -\pi_? +
\end{aligned}
$$

- The -293 term must be replaced by nearest $\pi_?$ (i.e., 271)
- If so, what is the net error of the prime continued fraction?
- Can you prove that the nearest integer converges faster (ROC)?
- Is this a generalizaton of a base N representation?
- What is the efficency (i.e., convergence) of base 2 vs. base π_k?

Real numbers $\mathbb{R}$ 7.3.4

- Once irrational numbers were accepted, *reals* $\mathbb{R}$ must coexist
- Stevin first introduces *finite* decimal fractions, i.e., $\mathbb{Q}$ 1585
- $\mathbb{R}$'s recognized when *convergence* are codified
  `http://www-history.mcs.st-and.ac.uk/HistTopics/Real_numbers_2.html`
  - $\mathbb{R}$'s were first defined between Hankel $\rightarrow$ Cantor c1870
  - Irrational numbers are a subset of reals
    - Integers are a subset of reals
    - Prime numbers are a special subset of integers
  - The details p. 526 must wait till Chap. 24 (Sets)
  - However today we all know well what they are: $\pi, e, \sqrt{\pi}$, etc.
  - This acceptance came slowly 16^{th}c
  - $\mathbb{R}$ may be ordered (e.g., $4>3$)

Complex $\mathbb{C}$ numbers

- Complex numbers $\mathbb{C}$, like $\mathbb{R}$s, were accepted late c1851
  - One can define a complex number without the concept of a real
  - Gaussian integers(e.g., $3+4j$) are an example
  - $\mathbb{C}$ cannot be ordered (e.g., Is $3+4i < 8-5i$?)
  - Which is larger $|3+4i|$ or $|8-5i|$?
- Matrix algebra of complex numbers

$$a \equiv \alpha + i\beta \leftrightarrow A \equiv \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

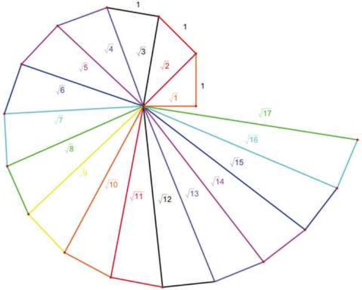
$$s \equiv \sigma + i\omega \leftrightarrow S \equiv \begin{bmatrix} \sigma & \omega \\ -\beta & \omega \end{bmatrix}$$

$$a \cdot s = (\alpha\sigma - \beta\omega) + i(\beta\sigma + \alpha\omega) \leftrightarrow A \cdot S = \begin{bmatrix} \alpha\sigma - \beta\omega & \beta\sigma + \alpha\omega \\ -(\beta\sigma + \alpha\omega) & \alpha\sigma - \beta\omega \end{bmatrix}$$

Lect NS 8.3 Stream 2: Geometry 8.3.1

- Euclid's Ruler and Compass constructions
  - Not restricted to integers!
  - So why were the Greeks so focused on integers?
- $\perp$ from point to line
- Bisection of angles (example)
- Circles and lines: basic geometry
- Conic sections
- This work is so much richer (easy) than integral relations (hard)
  - The use of integers placed a constraint on the problems, which drove people to fundamentals

Spiral of Theodorus:



rplayer.files.wordpress.com/2009/03/spiral-of-theodorus.png

- The n^{th} triangle has lengths $c = \sqrt{n+1}, b = \sqrt{n}, a = 1$
  - Thus $n + 1 = n + 1$ (since $c^2 = a^2 + b^2$)
- This spiral may be generated via a geometric argument

- Construction in terms of rectangles (vs spirals from previous figure)
- Page 46 of Stillwell

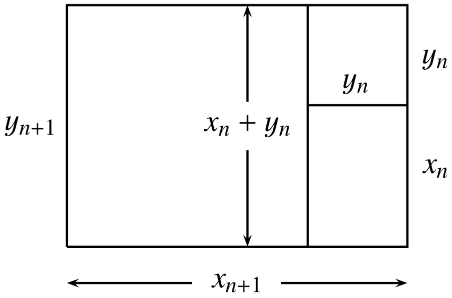


Figure: Starting from (y_n, x_n), recursively "grow" the square $y_n \times y_n$ into a rectangle.

Geometry and irrational numbers III

The *continued fraction expansion* has properties is similar to the GCD

- For example p. 48:

$$\phi = 1+\sqrt{2} = 2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}$$
$$= 2+1 \int 2+1 \int 2+1 \int \cdots$$

Fibonacci Sequence[4] 8.3.5

$$f_n = f_{n-1} + f_{n-2}$$

- This is a 2-sample *moving average* difference equation
- $f_n = [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \cdots]$, assuming $f_0 = 0, f_1 = 1$:
- Sol: (p. 194_3, 181_2): $\sqrt{5}\, f_n \equiv \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \to \left(\frac{1+\sqrt{5}}{2}\right)^n$
  - $\lim_{n\Rightarrow\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2}$
  - Ex: $34/21 = 1.6190 \approx \frac{1+\sqrt{5}}{2} = 1.6180$ 0.10% error
- Try Matlab's $rat(2 + sqrt(5)) = 4 + 1 \int 4 + 1 \int 4 + 1 \int 4 + \cdots$ p. 28

[4] https://en.wikipedia.org/wiki/Fibonacci_number

WEEK 4

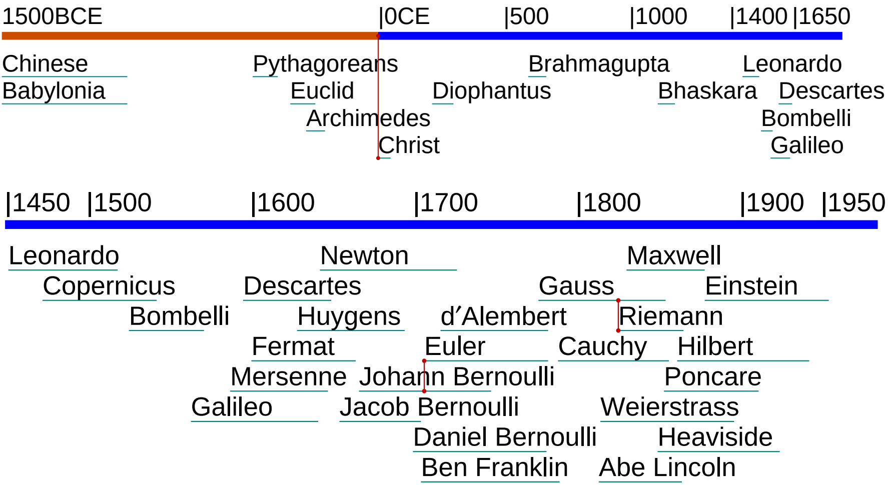
9 *Pell's Equation:* $n^2 - Nm^2 = 1$ (i.e., $y^2 = Nx^2 + 1$)
  The GCD solution
  The eigenvalue solution
  Extensions of *Pythagorean triplets*: $\{n, m; N\}$

10 Geometry & the Pythagorean Theorem

11 Review for Exam I

12 No Class (Exam I)

Mathematical Time Line 16-21 CE 1.1.3



Pell's Equation solution 9.4.1

- The solution to Pell's Eq ($N = 2$):

$$x^2 - Ny^2 = 1$$

- Solutions x_n, y_n are given by a recursion ($i = \sqrt{-1}$)

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = i \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

starting from the trivial solution $[x_0, y_0]^T = [1, 0]^T$.

- It follows that

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = i^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

- Then $x_n^2 - 2y_n^2 = 1$

Pell's Equation $N = 2$ 9.4.2

- Case of $N = 2$ & $[x_0, y_0]^T = [1, 0]$
  Note: $x_n^2 - 2y_n^2 = 1, \qquad x_n/y_n \to \sqrt{2}$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = +i \begin{bmatrix} 1 \\ 1 \end{bmatrix} = +i \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad -1^2 + 2 \cdot 1^2 = 1$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = -1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = -1 \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad 3^2 - 2 \cdot 2^2 = 1$$

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = -i \begin{bmatrix} 7 \\ 5 \end{bmatrix} = -i \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad -7^2 + 2 \cdot 5^2 = 1$$

$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = +1 \begin{bmatrix} 17 \\ 12 \end{bmatrix} = +1 \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 5 \end{bmatrix} \qquad 17^2 - 2 \cdot 12^2 = 1$$

$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = +i \begin{bmatrix} 41 \\ 29 \end{bmatrix} = +i \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 17 \\ 12 \end{bmatrix} \qquad -41^2 + 2 \cdot 29^2 = 1$$

- The general solution to Pell's Eq may be found by *eigenvalue analysis*
- The eigenvalues are given by

$$\det \begin{bmatrix} 1-\lambda & N \\ 1 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 - N = 0$$

$$(\lambda - 1)^2 = N$$

thus

$$\lambda_\pm = 1 \pm \sqrt{N}$$

- The solutions must be something like

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \beta^n \begin{bmatrix} 1 & N \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} \lambda_+^n \\ \lambda_-^n \end{bmatrix}$$

Simple expression for A^n by *Diagonalization* 9.4.4

- Matrix A may be written in *diagonal form* as follows:
  - Let [Check out the Matlab command $[E, Lambda] = eig(A)$]

$$E = \frac{1}{\sqrt{3}} \begin{bmatrix} -\sqrt{2} & \sqrt{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.8165 & -0.8165 \\ 0.5774 & 0.5774 \end{bmatrix}$$

  - This matrix has the following (unitary transformation) property

$$(E^{-1}AE)^n = \Lambda^n \equiv \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}^n = \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix}$$

- Inverting this expression gives a simple expression for A^n

$$A^n = \left(E\Lambda E^{-1}\right)^n = E \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} E^{-1}$$

- The general solution is then (x_n, y_n)

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = i^n \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = i^n \, E \begin{bmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{bmatrix} E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

  The relative "weights" on the eigenvalues are determined by (for $N = 2$)

$$E^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{det(E)} \begin{bmatrix} e_{22} & -e_{12} \\ -e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\sqrt{3}}{2\sqrt{2}} \begin{bmatrix} -1 & \sqrt{2} \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Show that
  - $x_n^2 - 2y_n^2 = 1$ (for $N = 2$)
  - x_n, y_n are complex integers

Composition solution to Pell's Equation 9.4.7

- Prove that the following (x, y) pair satisfy Pell's Equation

$$(x(t), y(t)) = \frac{(Nt^2+1, 2t)}{Nt^2-1}.$$

Proof:
$x^2 - Ny^2 = \frac{(Nt^2+1)^2 - N(2t)^2}{(Nt^2-1)^2} = \frac{(N^2t^4+2Nt^2+1)-4Nt^2}{(Nt^2-1)^2} = \frac{N^2t^4-2Nt^2+1}{(Nt^2-1)^2} = \frac{(Nt^2-1)^2}{(Nt^2-1)^2} = 1$

- As before let $t = p/q,\ p, q \in N$.

$$(x(p/q), y(p/q)) = \frac{q^2}{q^2}\frac{(N(p/q)^2+1, 2p/q)}{N(p/q)^2-1} = \frac{(Np^2+q^2, 2pq)}{Np^2-q^2}$$

- Thus in terms of $(\forall p, q$ such that $q \neq p\sqrt{N})$

$$x^2 - Ny^2 = \frac{(Np^2+q^2)^2 - 4Np^2q^2}{(Np^2-q^2)^2} = \frac{N^2p^4 - 2Np^2q^2 + q^4}{N^2p^4 - 2Np^2q^2 + q^4} = 1$$

- E.G. $p = q = 1, z^2 = x^2 - Ny^2 = \frac{N^2-2N+1}{(N-1)^2}$

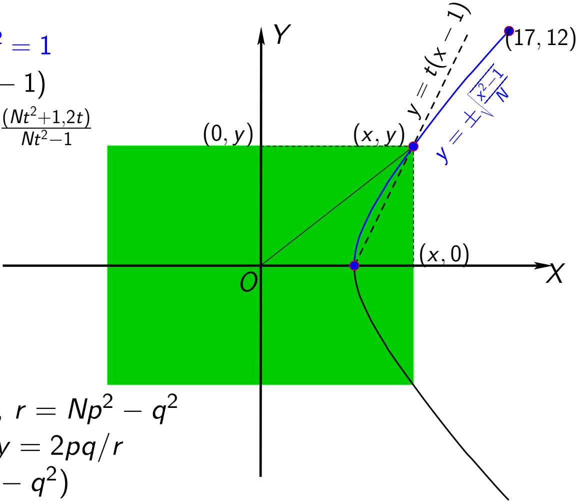
Proof I:

1) $x^2 - Ny^2 = 1$

2) $y = t(x-1)$

3) $(x,y) = \frac{(Nt^2+1, 2t)}{Nt^2-1}$

4) $t = p/q$



Choose $p, q, N \subset \mathbb{N}$, $r = Np^2 - q^2$

$x = (Np^2 + q^2)/r$, $y = 2pq/r$

$\tan(\theta) = 2qp/(Np^2 - q^2)$

Demo: `EvalPellEq.m`

Quadratic Equation solution

- Quadratic equation as you learned it (99%?)

$$ax^2 + bx + c = 0 \quad \rightarrow \quad x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- Derivation by completion of square: $\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$
- Expanding on the left reduces to the quadratic:

$$x^2 + bx + \cancel{\left(\frac{b}{2}\right)^2} = \cancel{\left(\frac{b}{2}\right)^2} - c$$

  - I'm suggesting you remember the green formula over the red one
- To do: Determine *radical* expressions for cubics and quartics?

Lect NS 11.1: Review of Number systems 11.5.1

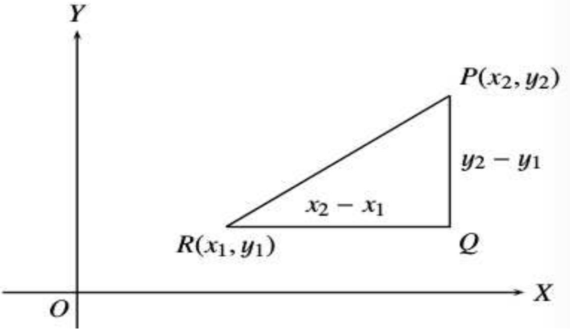
- Review of Homework solutions:
  - Pyth Triplets
  - Euclid's Formula
  - Euclid Alg
  - Continued fractions
  - Role of acoustics in Greek theory
  - Irrational numbers
  - Modern applications of number theory
  - Status of mathematics by 500 CE

Lect AE 12.5 *Stream 2:* Analytic Geometry

- Prepare for Exam I
- Introduction to algebraic systems via geometry
  - Composition of Polynomials: Descartes great discovery (p. 113_2)

Lect NS 10.4 Stream 2: Geometry

- Distance is related to length first defined as the "Euclidean length"



- Extended definitions of length require:
  - *line integrals* (i.e., calculus: $\int_a^b f(x) \cdot dx$) c1650
  - *Complex vector dot products* $||x||^2 = \sum_k x_k^2$, $||x - y||$
  - N-dimensional complex "Hilbert space," (i.e., "normed" vector spaces)

- *Schwartz inequality* is related to the shortest distance between a point and a line (is $\perp$)
  - Given two vectors U, V the $\perp$ may be found by minimizing the line from the end of one, to the other:

$$\min_{\alpha} ||V - \alpha U||^2 = ||V||^2 + 2\alpha V \cdot U + \alpha^2 ||U||^2 > 0$$

$$0 = \partial_\alpha \left(V - \alpha U\right) \cdot \left(V - \alpha U\right)$$

$$= V \cdot U - \alpha^* ||U||^2$$

$$\therefore \quad \alpha^* = V \cdot U / ||U||^2$$

  - The *Schwarz inequality* follows:

$$I_{\min} = ||V - \alpha^* U||^2 = ||V||^2 - \frac{|U \cdot V|^2}{||U||^2} > 0$$

$$0 \leq |U \cdot V| \leq ||U|| \; ||V||$$

Exam I 13.5.1

No class

Bibliography

John Stillwell. *Mathematics and its history; Undergraduate texts in Mathematics; 3d edition*. Springer, New York, 2002.